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Tensile fracture of doubly-convex cylindrical discs under diametral loading

K. G. PITT*, J. M. NEWTON
The School of Pharmacy, University of London, 29–39 Brunswick Square, London WC1N 1AX, UK

P. STANLEY
Department of Engineering, University of Manchester, Oxford Road, Manchester M13 9PL, UK

Doubly-convex cylindrical gypsum discs have been fractured under the action of two diametrically opposed in-plane forces. The disc diameter was constant throughout the test series. The ratio of cylinder length to diameter ranged from 0.06 to 0.30; the ratio of cylinder diameter to radius of curvature of the disk faces was varied from 0 to 1.43. The fracture loads obtained have been correlated with stress data obtained from the photoelastic analysis of Pitt *et al.* An empirical equation, valid for any brittle material, relating the tensile strength of the material to the fracture load and dimensions of a doubly-convex disc has also been developed.

Nomenclature

C, K	constants
D	diameter of disc
F	comparative factor
I_{max}	stress factor
P	load
P_s	fracture load of convex-faced disc
P_2	fracture load of plane-faced disc with $W/D = 0.2$
R	radius of face-curvature
t	overall thickness of convex-faced disc
T	thickness of plane-faced specimen of convex-faced disc
W	cylinder length
σ_f	tensile strength of material
σ_t	maximum tensile stress in convex-faced disc
σ_x	uniform tensile stress in plane-faced disc

1. Introduction

The simple plane-faced cylindrical disc specimen subjected to two diametrically opposed loads uniformly distributed along generators of the disc is widely used in the determination of the tensile fracture stress of brittle materials. The test was devised originally by two Brazilian engineers [1] and is referred to as the "Brazilian disc" test or the "indirect" tensile test, "indirect" because of the apparent anomaly of deriving the tensile fracture stress from compressive loading.

There is a complete analytical solution [2] for the elastic stress state induced in the plane-faced disc specimen by this form of loading. An important feature of this solution is that a uniform tensile stress is developed normal to and along the loaded diameter (i.e. the diameter joining the two loads). This uniform tensile stress, σ_x, is given by

$$\sigma_x = \frac{2P}{\pi DT} \qquad (1)$$

where P is the applied load, D is the diameter of the specimen, and T is the thickness. Using Equation 1, the tensile strength (i.e. tensile fracture stress) of the material is readily obtained from the fracture load.

The interpretation of fracture load data for non-plane-faced cylindrical discs (e.g. many pharmaceutical tablets), however, for which the stress distribution is unknown, is problematic. Such data have been obtained for an extensive series of doubly-convex gypsum disc specimens of differing face-curvature and cylinder length. In assessing the strength characteristics of these specimens, one approach is to regard the basic problem as a stress analysis problem, i.e. what are the stresses induced in the disc by two diametrically opposed forces uniformly distributed along generators of the cylindrical portion of the disc? The photoelastic method of stress analysis [3] has been used by Pitt *et al.* [4] in an assessment of the stress distribution in a range of convex-faced cylindrical discs subjected to diametral compression (see Fig. 1). These authors showed that the maximum tensile stress (σ_t) in a convex-faced disc subjected to diametral compression could be calculated from the expression

$$\sigma_t = \frac{2P}{\pi DW} I_{max} \qquad (2)$$

where P is the load, I_{max} is a stress factor (see later), D is the diameter of the disc, and W is the cylinder length (see Fig. 1). The use of this equation for the determination of the material tensile strength of the gypsum

*Present address: Pharmaceutical Development Laboratory, The Wellcome Foundation, Temple Hill, Dartford, DA1 5AM, Kent, UK.

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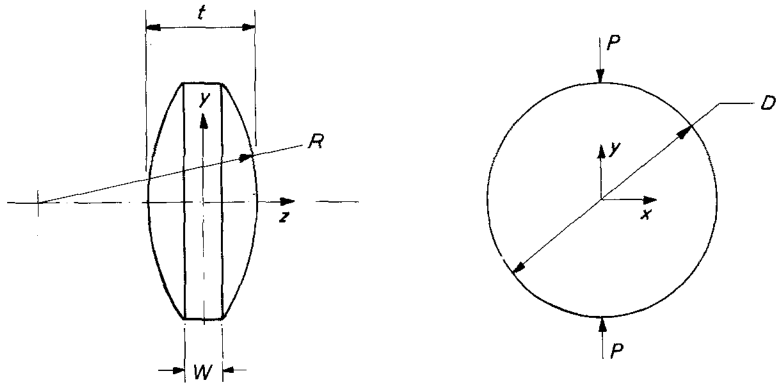

Figure 1 Side and front elevations of convex-faced disc showing axes and symbols.

from the fracture load values for the different disc specimens, is described in this paper.

An alternative approach, based on an analysis of the fracture load data, is also described. From this work a generally valid empirical equation is developed, relating the material tensile strength to the fracture load and dimensions of a doubly-convex disc.

2. Experimental details

2.1. Material

The brittle characteristics of the chosen test material, gypsum, are well known. Furthermore, it is relatively cheap, readily handled and available as a consistent high-quality product. In this work the gypsum was in the form of autoclaved plaster (α-calcium sulphate hemihydrate, Kaffir D, Cafferatta and Co.) of particle size 150 to 600 μm. In order to make the test specimens, the powder was blended with water, in the ratio 2.5 : 1, and cast into a closed steel mould to form twelve nominally identical discs. After an initial setting period of 30 min, the discs were removed from the mould and stored for 28 days at nominally constant conditions of room temperature and humidity before the fracture tests.

2.2. Specimen details

For a given diameter, the doubly-convex disc can be characterized in terms of two independent variables: (i) the cylinder length, W (see Fig. 1), and (ii) the face radius of curvature, R (see Fig. 1). In designing a test series, the principal consideration was to generate ranges for these variables realistically related to the ratios encountered in actual pharmaceutical tablets. The radii of curvature finally selected, non-dimensionalized with respect to diameter, are given in Table I, together with the corresponding conventional descriptions. The diameter in every case was 12.5 mm. Each face-curvature was studied in conjunction with cylinder length/diameter ratios (W/D) of 0.06, 0.1, 0.2 and 0.3.

TABLE I Face-curvature ratios of gypsum discs

Face-curvature ratio, D/R	Curvature description
0	Flat
0.25	Micro
0.50	Shallow
0.67	Normal
1.00	Unity
1.25	Deep
1.43	Coating

2.3. Fracture tests

The disc specimens were fractured between parallel hardened steel platens in a specially designed commercial test machine (CT40, Engineering Systems, Nottingham, UK, or Instron π-D, Instron, High Wycombe, Bucks, UK), using a platen closure rate of 1 mm min^{-1}. The bottom platen was kept stationary. For each combination of face-curvature and cylinder length, 20 specimens were fractured.

3. Experimental results

A summary of the mean tensile strengths of plane-faced discs ($D/R = 0$) of differing cylinder lengths, derived from Equation 1 using the respective mean fracture load, is presented in Table II. These results show that the determined tensile strengths are virtually identical (the differences are insignificant at probability levels $p > 0.05$), indicating that the plane stress assumption [5] is valid over this thickness range. It was concluded, therefore, that the manufacturing method and testing procedures were satisfactory for the assessment of the tensile strength of the convex-faced specimens.

A summary of the mean fracture loads, with coefficients of variation, for the convex-faced gypsum discs is given in Table III. The results show that, in general, the fracture loads increase with increasing face-curvature. The variability, as indicated by the coefficients of variation, shows a different trend. There is no marked dependence on face-curvature but the thinner discs ($W/D = 0.06$ and 0.1) have a higher coefficient of variation than the larger W/D specimens. These variability trends are consistent with the results of the photoelastic analysis of Pitt *et al.* [4], which demonstrate the pronouncedly non-uniform nature of the tensile stresses in the smaller cylinder length discs, and also with the predicted effect of increasing volume on strength [6].

TABLE II Mean tensile strengths of plane-faced gypsum discs. (Plaster to water ratio = 2.5 : 1, discs fractured after 28 d storage)

Cylinder length ratio, W/D	Tensile strength (MN m^{-2})	Coefficient of variation (%) $n = 20$
0.1	5.22	18.7
0.2	5.25	10.1
0.3	5.54	13.9

TABLE III Mean fracture loads (P_s) of convex-faced gypsum discs

Face-curvature ratio, D/R	Cylinder length ratio, W/D							
	0.06		0.10		0.20		0.30	
	P_s (kg)	c.v. (%)*	P_s (kg)	c.v. (%)	P_s (kg)	c.v. (%)	P_s (kg)	c.v. (%)
0.00	6.1	22.7	12.8	18.6	26.8	10.1	40.8	13.9
0.25	7.5	20.5	17.7	5.5	33.8	10.8	53.1	10.6
0.50	9.6	18.6	18.8	13.3	35.4	14.4	52.6	9.6
0.67	10.9	20.6	19.6	13.3	40.0	12.8	57.2	9.1
1.00	11.9	19.5	20.6	14.4	45.3	9.5	64.3	8.0
1.25	16.1	21.4	27.3	14.2	47.5	12.2	65.8	14.8
1.43	17.9	15.2	28.0	14.8	52.6	11.9	72.0	9.4

*c.v. = coefficient of variation (n = 20).

TABLE IV Comparison of actual and predicted mean fracture loads for convex-faced gypsum discs

Cylinder length ratio, W/D	Face-curvature ratio, D/R	Actual fracture load (kg)	Predicted fracture load (kg)
0.10	0.	12.8	12.9
0.10	0.50	18.8	15.4
0.10	1.00	20.6	14.2
0.10	1.25	27.3	30.7
0.10	1.43	28.0	51.5
0.06	1.43	17.9	24.9
0.20	1.43	52.6	56.9
0.30	1.43	72.0	60.1

4. Discussion

In Table IV the mean fracture loads of eight of the convex-faced gypsum discs are compared with the predicted fracture loads derived from the photoelastic stress factors of Pitt *et al.* [4] (see Table V). These factors are defined as the ratio of the maximum tensile stress in a convex-faced disc to that in a plane-faced disc of the same diameter and cylinder length subjected to the same load. The predicted fracture loads are calculated from Equation 2, assuming that the specimen material has a tensile strength of 5.25 $\mathrm{MN\,m^{-2}}$ (see later) and that a disc will, therefore, fracture at the load for which the stress reaches that value. (Details of the stress distribution in each disc specimen can be obtained from the account of the photoelastic work [4].) The comparison shows that for most discs there is a reasonable correlation between the actual and predicted fracture loads.

In the case of the disc with the greatest face-curvature (D/R = 1.43) and longest cylinder length (W/D = 0.3), the observed and predicted fracture loads were 72.0 and 60.1 kg, respectively. For the disc with the same face-curvature but a smaller cylinder length ratio of 0.1, the observed fracture load was considerably less than that predicted. These differences between observed and predicted fracture loads are probably due to slight differences in the stress states in the gypsum discs and the photoelastic specimens resulting from differences in the contact zone deformation in the two materials, and to the lack of perfect homogeneity and isotropy in the gypsum discs.

The photoelastic results [4] are clearly relevant in correlating experimentally determined stresses with observed fracture loads in geometrically similar brittle discs, but the correlation covered only eight specific combinations of face-curvature and cylinder length. The development of some means of reliably predicting material tensile fracture stress from the fracture loads of discs over a wide range of face-curvatures and cylinder lengths was required. The photoelastic data offered some scope for interpolation and extrapolation, but this scope was limited. However, the results of the gypsum disc fracture tests were adequate for this purpose; this development of the work is described below.

The variability in the fracture load of plane-faced discs with W/D = 0.2 was somewhat smaller than that of the other discs (see Table II). For this reason

TABLE V Photoelastic stress factors, I_{max} [4]

Cylinder length ratio, W/D	Face-curvature ratio, D/R	Stress factor, I_{max}
0.10	0.	1.00
0.10	0.50	0.84
0.10	1.00	0.91
0.10	1.25	0.42
0.10	1.43	0.25
0.06	1.43	0.31
0.20	1.43	0.45
0.30	1.43	0.65

TABLE VI Comparative factors, F, for convex-faced discs

Face-curvature ratio, D/R	t/W	Mean fracture load (kg)	Factor ($F = P_s/P_2$)	Factor (from Eqn. 10)	Percentage difference in factors
Cylinder length = 0.06 W/D					
0.	1.00	6.11	0.228	0.243	−6.75
0.25	2.04	7.49	0.279	0.290	−3.79
0.50	3.08	9.62	0.359	0.336	6.47
0.67	3.80	10.92	0.407	0.368	9.65
1.00	5.31	11.86	0.442	0.435	1.67
1.25	6.32	16.12	0.601	0.479	20.20
1.43	7.93	17.93	0.669	0.551	17.60
Cylinder length = 0.1 W/D					
0.	1.00	12.81	0.477	0.483	−1.26
0.25	1.63	17.73	0.661	0.582	12.01
0.50	2.25	18.81	0.701	0.680	2.97
0.67	2.68	19.62	0.729	0.748	−2.67
1.00	3.58	20.64	0.768	0.891	−16.05
1.25	4.19	27.20	1.016	0.987	2.82
1.47	5.16	28.03	1.044	1.140	−9.22
Cylinder length = 0.2 W/D					
0.	1.00	26.82	1.000	1.082	−8.20
0.25	1.31	33.75	1.258	1.220	2.63
0.50	1.62	35.36	1.318	1.357	−3.02
0.67	1.84	40.03	1.493	1.453	2.66
1.00	2.29	45.33	1.690	1.653	2.19
1.25	2.60	47.49	1.771	1.787	−0.93
1.43	3.08	52.62	1.962	2.000	−2.00
Cylinder length = 0.3 W/D					
0.	1.00	40.79	1.521	1.681	−10.52
0.25	1.21	53.08	1.979	1.832	7.43
0.50	1.42	52.55	1.959	1.981	−1.13
0.67	1.56	57.19	2.132	2.087	2.08
1.00	1.86	64.32	2.398	2.306	3.82
1.25	2.06	65.83	2.454	2.453	0.02
1.43	2.39	69.55	2.606	2.687	−3.14

the tensile fracture stress of the gypsum material used in this further analysis was taken to be that of the plane-faced disc specimen with $W/D = 0.2$, i.e. $5.25\,\mathrm{MN\,m^{-2}}$. Assuming this value, a comparative factor was obtained which allowed the tensile strength of a material, σ_f, to be calculated from the fracture load of a convex-faced disc specimen. This comparative factor, F, was defined as the ratio of the mean fracture load of the convex-faced disc, P_s, to that, P_2, of the plane-faced disc with $W/D = 0.2$. The values of the factor are tabulated in Table VI and are used as follows. By definition

$$F = \frac{P_s}{P_2} \tag{3}$$

and therefore

$$P_2 = \frac{P_s}{F} \tag{4}$$

Referring to Equation 1, and using the results for the plane-faced disc with $W/D = 0.2$, it follows that

$$\sigma_f = \frac{2P_2}{\pi DW} = \frac{10P_2}{\pi D^2} \tag{5}$$

i.e. that

$$\sigma_f = \frac{10P_s}{\pi D^2 F} \tag{6}$$

The tensile strength of a brittle material can be readily calculated from the mean fracture load of a convex-faced disc specimen using Equation 6, with the appropriate F value. For example, the material tensile strength of a deep curvature ($D/R = 1.43$) disc of cylinder length $W/D = 0.3$, is obtained from the fracture load using an F value of 2.61 (see Table VI). The calculation is valid for specimens of the relevant dimensions failing in tension.

In an attempt to develop a relationship between the comparative factor F and the shape parameters of the

TABLE VII Linear regression analysis parameters for the comparative factor, F, and the ratio, t/W

Cylinder length ratio, W/D	Correlation coefficient	Probability*	Gradient (Fig. 2)	Intercept (Fig. 2)
0.06	0.9843	0.58×10^{-4}	0.065	0.154
0.10	0.9580	0.68×10^{-3}	0.130	0.389
0.20	0.9895	0.25×10^{-4}	0.448	0.620
0.30	0.9597	0.61×10^{-3}	0.727	0.956

*Probability = probability that there is no linear relationship.

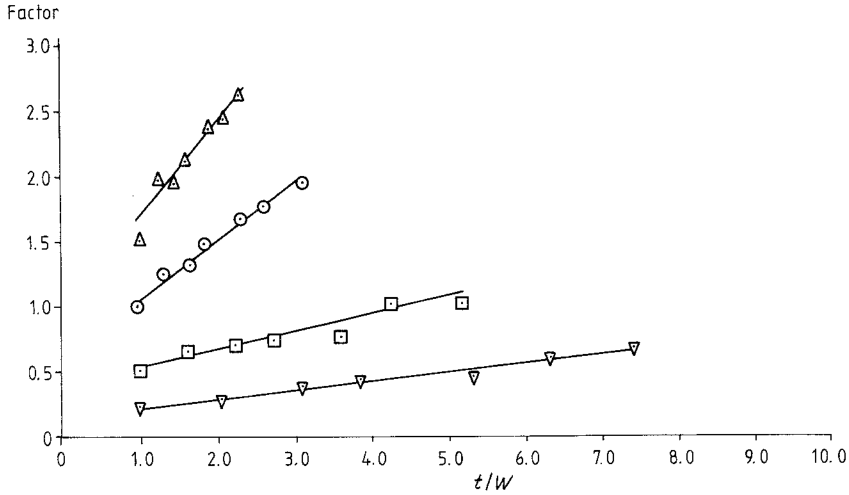

Figure 2 Comparative factor, F, plotted against t/W for convex-faced discs. W/D = (▽) 0.06, (□) 0.1, (○) 0.2, (△) 0.3.

convex-faced discs, and following an indication from the photoelastic work [4], linear regression analyses were undertaken between the factor and the reciprocal of the parameter t/W (see Table VI), t being the overall thickness of the disc (see Fig. 1). (It is readily shown that $t = W + 2R - (4R^2 - D^2)^{1/2}$). The correlation coefficients and related probabilities for these regression analyses are presented in Table VII. The plot of the factors against t/W for each of the four W/D values in the test series is illustrated in Fig. 2. These results indicate that for discs of a given cylinder length and varying face-curvature there is a good correlation between the parameter t/W and the comparative factor, F. The linear relationship is highly significant for all cases, and demonstrates that the parameter t/W could be of practical use in the determination of the tensile strength of a brittle material from the fracture load of convex-faced disc specimens. Fig. 2 shows that for a given W/D, within the range covered in the tests, the factor, F, is proportional to t/W, i.e.

$$F = K\frac{t}{W} + C \qquad (7)$$

where K and C are parameters dependent upon W/D.

The functional relationships between (a) K and W/D, and (b) C and W/D were found as follows.

(a) The gradients (see Table VII) of the lines in Fig. 2 (i.e. K) were plotted against W/D to give an approximately linear relationship (see Fig. 3). This is referred to as the "first function" of W/D and is given in Fig. 3.

(b) The intercepts (see Table VII) of the lines in Fig. 2 with the vertical axis (i.e. C) were plotted against W/D, giving again an approximately linear relationship (see Fig. 4). This is referred to as the "second function" of W/D and is given in Fig. 4.

Equation 7 for the comparative factor, F, becomes

$$F = (\text{first function of } W/D)(t/W) + \text{second function of } W/D \qquad (8)$$

giving, from Figs 3 and 4

$$F = (2.84W/D - 0.126)(t/W) + (3.15W/D + 0.01) \qquad (9)$$

or

$$F = 2.84t/D - 0.126t/W + 3.15W/D + 0.01 \qquad (10)$$

Using this form of F, Equation 6 for determining the material tensile strength from the fracture load of a convex-faced disc may be written in terms of the disc dimensions as

$$\sigma_f = \frac{10P_s}{\pi D^2}\left(2.84\frac{t}{D} - 0.126\frac{t}{W} + 3.15\frac{W}{D} + 0.01\right)^{-1} \qquad (11)$$

In Table VI the values of F calculated from

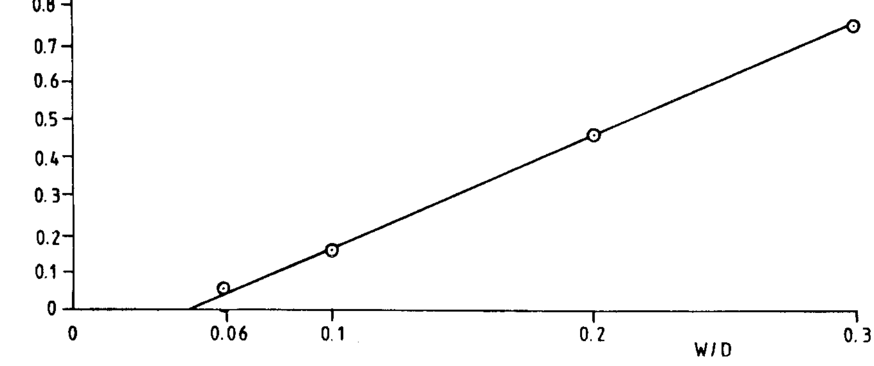

Figure 3 K plotted against W/D, giving first function of W/D.

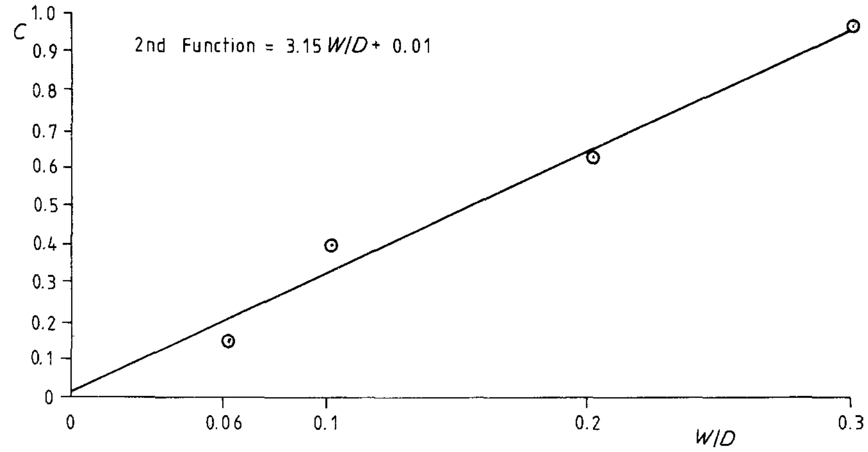

Figure 4 C plotted against W/D, giving second function of W/D.

Equations 3 and 10 are compared. The percentage difference is generally in the $\pm 10\%$ band for cylinder lengths in the range $0.06 \leqslant W/D \leqslant 0.3$, except at the lower limit of this range when the face-curvature ratio is relatively large ($D/R > 1.0$), in which cases the differences may be 20%. Thus, Equation 11 allows the tensile strength of a brittle material to be calculated reasonably accurately from a knowledge of the fracture load of a convex-faced disc, for cylinder length ratios $0.1 \leqslant W/D \leqslant 0.3$ and for discs with a cylinder length ratio of $W/D = 0.06$ and face-curvature ratio $D/R < 1.0$.

5. Conclusions

The fracture loads of a variety of convex-faced gypsum discs have been determined and found to correlate with those predicted from the photoelastic results of Pitt *et al.* [4].

The tensile strength of a brittle material can be assessed from the fracture load of a convex-faced disc specimen, diameter D, by means of the following equation

$$\sigma_f = \frac{10P_s}{\pi D^2 F}$$

where the factor, F, is dependent upon the cylinder length ratio and face-curvature ratio of the specimen and can be determined directly as the ratio of fracture loads (see Table VI) or more generally from the expression

$$F = 2.84t/D - 0.126t/W + 3.15W/D + 0.01$$

Both of these expressions are applicable to discs which fail in tension when loaded diametrically.

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